REGULARITY AND RESOLUTIONS FOR MULTIGRADED MODULES

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Abstract. This paper is concerned with the relationships between two concepts, vanishing of cohomology groups and the structure of free resolutions. In particular, we study the connection between vanishing theorems for the local cohomology of multigraded modules and the structure of their free multigraded resolutions.

1. Introduction

Castelnuovo-Mumford regularity was first defined by Mumford [15] for coherent sheaves $\mathcal{F}$ on projective spaces $\mathbf{P}^n$. $\mathcal{F}$ is m-regular if

$$H^i(\mathbf{P}^n, \mathcal{F}(j)) = 0, \quad \forall \ \ i+j \geq m, \quad \forall \ \ i \geq 1.$$

Mumford showed that the regularity of a sheaf measures the smallest degree twist for which the sheaf is generated by its global sections. More generally, regularity of a sheaf has implications for the structure of a resolution of that sheaf by direct sums of the sheaves $\mathcal{O}(j)$.

There is a corresponding notion for graded modules. Regularity of $\mathbb{Z}$-graded modules was investigated by many people, notably, Bayer-Mumford [1], Bayer-Stillman [2], Eisenbud-Goto [7], and Ooishi [17]. Let $S = K[x_1, \ldots, x_n]$ be the polynomial algebra in n variables over a field K with the standard grading. If M is a finitely generated graded S-module, $B = (x_1, ..., x_n)$ is the maximal ideal, then M is m-regular if

$$H^i_B(M)_j = 0, \quad \forall \ \ i+j \geq m+1.$$

An important result is:

Theorem 1.1. *([1]) Suppose K is a field and M is a graded S-module. Then M is m-regular if and only if the minimal free graded resolution of M has the form*

$$0 \to \bigoplus_{\alpha=1}^{r_s} S(-d_{\alpha,s}) \to \cdots \to \bigoplus_{\alpha=1}^{r_1} S(-d_{\alpha,1}) \to \bigoplus_{\alpha=1}^{r_0} S(-d_{\alpha,0}) \to M \to 0$$

where $d_{\alpha,i} \leq m+i$ for all $i \geq 0$.

This result gives an explicit relationship between the regularity of a graded module, in other words the vanishing of certain graded pieces of local cohomology $H^i_B(M)$, and degree bounds on generators of the free graded resolution of M.

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Recently, there has been a great interest in studying regularity of multigraded modules over a multigraded algebra. Geometrically such algebras arise as the coordinate algebras of projective toric varieties. Let S be a G-graded K-algebra where K is a field and G is a finitely generated abelian group. Let B be a monomial ideal in S and M a finitely generated G-graded S-module. Multigraded regularity relates the vanishing of $H^i_B(M)_{\mathbf{d}}$ for $\mathbf{d}$ in certain convex regions of G with the structure of resolutions of M. In case (S, B) is the homogeneous coordinate ring and irrelevant ideal of a smooth projective toric variety X, G is the divisor class group of X, and the geometric version of the theory imposes vanishing conditions on $H^i(X, \mathcal{F}\otimes\mathcal{O}(\mathbf{d}))$. A special case of this, with $G = \mathbb{Z}^2$, was done by the authors [12], and the general case is due to Maclagan and Smith [13]. Here we introduce the definition of multigraded regularity as in [13].

Let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_l\}$ with $\mathbf{c}_i \in G$ be a fixed subset of G, and let $\mathbb{N}\mathcal{C}$ be the sub-semigroup of G generated by $\mathcal{C}$ ($\mathbb{N} = \mathbb{Z}_{\geq 0}$). Define

$$\mathbb{N}\mathcal{C}[j] = \bigcup_{\mathbf{w}=(w_1,\dots,w_l)\in\mathbb{N}^l,\ \sum w_i = |j|} (\mathrm{sign}(j)(w_1\mathbf{c}_1 + \cdots + w_l\mathbf{c}_l) + \mathbb{N}\mathcal{C}). \tag{1.1}$$

Then M is $\mathbf{m}$-regular with respect to B and $\mathcal{C}$, if for each $i \geq 0$:

$$H^i_B(M)_{\mathbf{d}} = 0, \ \ \forall \mathbf{d} \in \mathbf{m} + \mathbb{N}\mathcal{C}[1-i], \tag{1.2}$$

and we denote this by $\mathbf{m} \in \mathrm{reg}_{B,\mathcal{C}}(M)$. There is no loss of generality in assuming that $S = K[x_1, ..., x_n]$ is a free algebra. The degrees of the generators x_i are elements $\mathbf{a}_i \in G$ and we let $\overline{\mathbf{a}}_i$ be their images in $G \otimes_{\mathbb{Z}} \mathbb{R}$. The set $\overline{\mathcal{A}} = \{\overline{\mathbf{a}}_1, ..., \overline{\mathbf{a}}_n\}$ is an integral vector configuration in $G \otimes_{\mathbb{Z}} \mathbb{R}$. One assumes

1. The monomial ideal B corresponds to a maximal cell $\Gamma \subset G \otimes_{\mathbb{Z}} \mathbb{R}$ in the vector configuration $\overline{\mathcal{A}}$ via
$$B := \langle \prod_{i\in\sigma} x_i : \sigma \subseteq [n] \text{ with } \Gamma \subseteq \mathrm{pos}(\overline{\mathbf{a}}_i : i \in \sigma\rangle,\ [n] := \{1, ..., n\}\rangle$$
where the positive hull of a set of vectors is
$$\mathrm{pos}(\mathbf{v}_1, ..., \mathbf{v}_m) := \{\lambda_1\mathbf{v}_1 + ... + \lambda_m\mathbf{v}_m : \lambda_i \in \mathbb{R}_{\geq 0}.\}$$
The choice of this maximal cell Γ is implicit in all the results of [13].
2. $\mathrm{pos}(\overline{\mathcal{A}})$ is a pointed cone with $\overline{\mathbf{a}}_i \neq \mathbf{0}$ for all i.

These two conditions are satisfied by (S, B), the homogeneous coordinate ring and irrelevant ideal of a projective toric variety. For the main results connecting to the structure of resolutions one assumes

3. $\mathcal{C} \subseteq \mathcal{K}$ where $\mathcal{K}$ is the Kaehler cone (for detailed definition see [13, Section 2.]).

Among their results are certain bounds on the generators and syzygies of an $\mathbf{m}$-regular module, but these are not as sharp as those of the classical theory. In particular they obtain, under the further hypothesis,

4. $\mathbb{N}\mathcal{C} = \mathcal{K} = \mathcal{K}^{\mathrm{sat}}$

not a free resolution, but a complex with B-torsion homology augmenting to the module M (see [13, Theorem 7.8]). Since modules with B-torsion homology become zero when converted to sheaves on the corresponding toric variety, this is adequate for applications for sheaf theory. But it is still of interest to obtain more precise results about the structure of free resolutions. In [12] a concept of “strong

regularity" was introduced which gave results analogous to Theorem 1.1. One of the themes of this paper is to study the effect of stronger regularity conditions on the structure of free resolutions.

We first investigate the relationship between certain regions

$$\mathcal{D}^i_B(J) \subset \mathcal{Z}^i_B(M) = \{\mathbf{d} \in G : H^i_B(M)_\mathbf{d} = 0\}$$

and the sets $J = \{J_0, J_1, \ldots, J_s\}$ in a minimal free resolution

$$0 \to \bigoplus_{\mathbf{d}_s \in J_s} S(-\mathbf{d}_s) \to \cdots \to \bigoplus_{\mathbf{d}_0 \in J_0} S(-\mathbf{d}_0) \to M \to 0.$$

of a G-graded S-module M. This question has two aspects. On the one hand, starting with J, one would like to find easy to describe regions $\mathcal{D}^i_B(J)$ with the property that any module M with a resolution of this type with degrees in J has $H^i_B(M)_\mathbf{d} = 0$ for $\mathbf{d} \in \mathcal{D}^i_B(J)$. On the other hand, starting from known regions $\mathcal{D}^i \subset G$ of vanishing of $H^i_B(M)_\mathbf{d}$ one would like to find a free resolution for M with the p-th syzygy module having degrees in regions J_p which can be described explicitly in terms of the regions $\mathcal{D}^i$. We will discuss this relationship in Section 3 after we introduce some necessary tools in Section 2.

In practice one would like the regions $\mathcal{D}^i_B(J)$ to be the intersection of G with a rational convex polyhedral region in $G \otimes \mathbb{R}$. The faces of these would be defined by linear inequalities. Thus one is led to consider regularity with respect to regions defined by $h \geq 0$ where h is a linear form. In effect this means that one is coarsening the multigrading to a simple, but generally nonstandard grading. This has been considered in a recent paper by Sidman-Van Tuyl-Wang [19] who extended some of the techniques used in [12]. They introduced a notion of positive coarsening vector which coarsens the multidegrees on M to a single nonstandard $\mathbb{Z}$ grading, and obtain finite bounds on the multidegrees of a minimal free resolution of M. If $\phi : G \to G_0$ is a surjection, we define a G_0-grading of M by

$$M_{\mathbf{d}_0} = \bigoplus_{\mathbf{d} \in \phi^{-1}(\mathbf{d}_0)} M_\mathbf{d}.$$

One considers vanishing regions contained in G_0 for $H^i_{B_0}(M)$ with respect to a G_0-graded ideal B_0, and a set $\mathcal{C}_0 \subseteq G_0$. The simplest case is when $G_0 = \mathbb{Z}$, and if $G = \mathbb{Z}^r$, $\phi(\mathbf{d}) = \mathbf{v} \cdot \mathbf{d}$ for a coarsening vector $\mathbf{v} \in G$. This case is studied in detailed in [19]. Often, the $(B, \mathcal{C})$-regular region can be estimated in terms of $(B_0, \mathcal{C}_0)$-regularity regions, defined by various ϕ's, and the advantage of considering the ϕ's is that one can often give easily described constraints on the regions J_p. We will discuss this in Section 4.

Another approach is to consider regularity with respect to a family of $(B_0, \mathcal{C}_0)$ defined by natural geometric conditions. One example we consider is the canonical decomposition of the nullcone:

$$Z = \mathbb{V}(B) = \bigcup_{i=1}^{t} Z_i, \quad Z_i = \mathbb{V}(B_i),$$

where (S, B) is the homogeneous coordinate ring and irrelevant ideal of a simplicial toric variety $\mathbf{P}_\Sigma$ defined by the fan Σ. The ideals B_i are linear. We employ a Mayer-Vietoris spectral sequence in local cohomology to relate vanishing regions of $H^i_B(M)$ to those of $H^i_{B_I}(M)$ with respect to ideals B_I, where B_I is defined by the intersection of the various Z_i. This method works especially well for fans of

the form $\Sigma = \Sigma_1 \times \cdots \times \Sigma_s$. In Section 5, we will discuss this issue, and also we illustrate its use for multiprojective spaces $\mathbf{P}^{m_1} \times \cdots \times \mathbf{P}^{m_t}$.

The example of multiprojective spaces shows the necessity of considering gradings ϕ which assign $\mathbf{0}$ to some of the variables, which was not treated in [19]. We can include such gradings by writing $S = R[x_j]$ where $R = K[x_i]$ where x_i are the variables with $\phi(\deg(x_i)) = 0$, and x_j are the variables that are ϕ-positive, i.e., admissible for the theory of regularity. For this, we need to generalize the theory developed in [13] to G-graded algebras over a commutative Noetherian ring R, not just a field. We do this in Section 6 following ideas of Ooishi [17]. We end this paper by giving some illustrative examples in Section 7.

2. Tools

This section recalls some well-known spectral sequences and derives some corollaries needed later.

Let $\mathcal{A}$ be an abelian category. We consider chain complexes, indexed by $\mathbb{Z}$, over $\mathcal{A}$ both of homological type:

$$C_\bullet = \cdots \longrightarrow C_n \xrightarrow{\partial} C_{n-1} \longrightarrow \cdots,$$

and of cohomological type:

$$C^\bullet = \cdots \longrightarrow C^n \xrightarrow{d} C^{n+1} \longrightarrow \cdots.$$

These are formally equivalent by defining $C^n = C_{-n}$. We let $C[p]^\bullet$ be the shifted complex: $C[p]^n = C^{p+n}$, and we often identify an object M in $\mathcal{A}$ with a complex concentrated in degree 0.

Given complexes $A^\bullet$, $B^\bullet$, the first bounded above and the second bounded below, we recall that the groups $\mathrm{Ext}^i(A^\bullet, B^\bullet)$, which exist if $\mathcal{A}$ has enough injective objects. This is computed by taking a quasi-isomorphism $B^\bullet \cong I^\bullet$ where $I^\bullet$ is a bounded below complex of injective objects. Then

$$\mathrm{Ext}^i(A^\bullet, B^\bullet) = H^i(\mathrm{Hom}^\bullet(A^\bullet, I^\bullet))$$

where $\mathrm{Hom}^\bullet(A^\bullet, I^\bullet)$ is the total hom functor defined in [11, pp. 63-64].

Proposition 2.1. *There is a spectral sequence*

$$E_1^{p,q} = \mathrm{Ext}^q(A^\bullet, B^p) \Longrightarrow \mathrm{Ext}^{p+q}(A^\bullet, B^\bullet).$$

This converges if $A^\bullet$ is bounded above and $B^\bullet$ is bounded below.

Proof. This is the hypercohomology spectral sequence of the functor $T = \mathrm{Hom}^\bullet(A^\bullet, -)$ applied to the complex $K = B^\bullet$ with the filtration $F = \sigma_{\geq p}$, whose graded terms are $B^p[p]$, [5, pp. 20-21]. □

Corollary 2.2. *If*

$$\cdots \longrightarrow C_2 \longrightarrow C_1 \longrightarrow C_0 \xrightarrow{\varepsilon} M \longrightarrow 0$$

is a resolution, then there is a convergent spectral sequence

$$E_1^{-p,q} = \mathrm{Ext}^q(A^\bullet, C_p) \Longrightarrow \mathrm{Ext}^{q-p}(A^\bullet, M).$$

Proof. By assumption, the module M, regarded as a complex concentrated in degree 0, is quasi-isomorphic with the complex $C_\bullet$, and we may thus replace M by that complex in the computation of Ext. Apply the proposition and shift to upper numbering in indexing: $p \to -p$. □

Proposition 2.3. *There is a spectral sequence*

$$E_1^{-p,q} = \mathrm{Ext}^q(A^p,\, B^\bullet) \Longrightarrow \mathrm{Ext}^{q-p}(A^\bullet,\, B^\bullet).$$

This converges if $A^\bullet$ is bounded above and $B^\bullet$ is bounded below.

Proof. We let $K^\bullet = \mathrm{Hom}^\bullet(A^\bullet,\, I^\bullet)$. Since $I^\bullet$ is a complex of injective objects, the functor $C^\bullet \to \mathrm{Hom}^\bullet(C^\bullet,\, I^\bullet)$ is exact. Thus we can define a decreasing filtration by

$$F^pK^\bullet = \mathrm{Hom}^\bullet(A^\bullet/\sigma_{\geq 1-p}A^\bullet,\, I^\bullet).$$

Moreover,

$$\mathrm{Gr}_F^p K^\bullet = \mathrm{Hom}^\bullet(\mathrm{Gr}_\sigma^{-p}A^\bullet,\, I^\bullet) = \mathrm{Hom}^\bullet(A^{-p}[p],\, I^\bullet) = \mathrm{Hom}^\bullet(A^{-p},\, I^\bullet)[-p]$$

The spectral sequence of a filtered complex

$$E_1^{-p,q} = \mathrm{H}^{q-p}(\mathrm{Gr}_F^{-p}K^\bullet) \Longrightarrow \mathrm{H}^{q-p}(K^\bullet)$$

gives our spectral sequence, in view of

$$\begin{aligned} \mathrm{H}^{q-p}(\mathrm{Gr}_F^{-p}K^\bullet) &= \mathrm{H}^{q-p}(\mathrm{Hom}^\bullet(A^p,\, I^\bullet)[p] \\ &= \mathrm{H}^q(\mathrm{Hom}^\bullet(A^p,\, I^\bullet)) = \mathrm{Ext}^q(A^p,\, B^\bullet). \end{aligned}$$

□

Let X_* be a simplicial topological space, that is, a contravariant functor from the category Δ whose objects are the totally ordered sets $[n] = \{1, ..., n\}$ and whose morphisms preserve the order. An augmented simplicial topological space $a : X_* \to S$ is of cohomological descent if the canonical map of functors from $\mathrm{D}^+(S) \to \mathrm{D}^+(S)$:

$$\varphi : Id \to \mathrm{R}a_*a^*$$

is an isomorphism (see [6, §5]). We will be interested in the following special case: Let $j : Z \hookrightarrow X$ be the inclusion of a closed subspace, and

$$Z = \bigcup_{i=1}^{t} Z_i$$

is a union of closed subsets. This defines an augmented simplicial space $a : W_* \to Z$ by taking the usual Cech complex:

$$W_k = \coprod_{\#I=k+1} W_I, \quad W_I = \bigcap_{i\in I} Z_i, \quad I \subseteq \{1, \ldots, t\}$$

Since $\coprod Z_i \to Z$ is a proper surjective map, a is of universal cohomological descent ([6, §5]). Let $a_k : W_k \to Z$ be the canonical map. The components of this, $a_I : W_I \to Z$ for $\#I = k+1$, are closed imbeddings. Thus, a_{k*} is an exact functor on the category of abelian sheaves, and hence $\mathrm{R}^i a_{k*} = 0$, for $i > 0$. The means that, for any abelian sheaf F on Z,

$$F \longrightarrow C^*(W_*, F)$$

is a resolution, where

$$C^k(W_*, F) = a_{k*}a_k^*F = \bigoplus_{\#I=k+1} a_{I*}a_I^*F$$

with the usual Cech differential ([6, §5]). Since j_* is an exact functor on abelian sheaves, we obtain a resolution

$$j_*F \longrightarrow j_*C^*(W_*, F) \quad \text{with} \quad j_*C^k(W_*, F) = \bigoplus_{\#I=k+1} F_{W_I,X}$$

where $F_{W_I,X}$ is the sheaf on X that coincides with $F \mid W_I$ on $W_I \subset X$ and is extended by 0 on the rest of X.

Apply this to the constant sheaf of integers $F = \mathbb{Z}_Z$. Recall that, for any closed subset $Z \subset X$ we have a canonical isomorphism:

$$H^i_Z(X,\, M) = \mathrm{Ext}^i(\mathbb{Z}_{Z,X},\, M)$$

where the left-hand side is local cohomology, and the right-hand side is the Ext group for the category of abelian sheaves on X ([9, Exposé I, Prop. 2.3]).

Proposition 2.4. *Let $Z = \bigcup_{i=1}^t Z_i$ be a union of closed subsets $Z_i \subset X$. Let M be a sheaf of abelian groups on X. There is a convergent spectral sequence*

$$E_1^{-p,q} = \bigoplus_{\#I=p+1} H^q_{W_I}(X,\, M) \Longrightarrow H^{q-p}_Z(X,\, M).$$

Proof. The local cohomology is computed via the Ext's as explained. Since $\mathbb{Z}_{Z,X}$ is quasi-isomorphic to the complex $j_*C^*(W_*, \mathbb{Z}_Z)$ we may replace that sheaf by that complex and apply Proposition 2.3. We then obtain the spectral sequence in the indicated form. □

When $t = 2$ this spectral sequence reduces to the Mayer-Vietoris sequence in local cohomology ([10, p. 212, Ch. III, Ex. 2.4]).

In our application, $X = \mathrm{Spec}(S)$ where S is a finitely generated G-graded R-algebra, with R being a commutative Noetherian ring, and G is a finitely generated abelian group. Equivalently, we can think of X as an affine scheme with an action of the group scheme $T = \mathrm{Spec}\, R[G]$. The most important case is when $G = \mathbb{Z}^r$ is free, then $T \cong \mathbf{G}_m^r$, where $\mathbf{G}_m$ is the multiplicative group. We are interested in G-graded S-modules M. They define quasicoherent sheaves $\tilde{M}$ on X also with a T-action.

We will be considering

$$Z = \mathbb{V}(B) = \bigcup_{i=1}^t \mathbb{V}(B_i) = \bigcup_{i=1}^t Z_i$$
$$W_I = \bigcap_{i\in I} Z_i = \mathbb{V}(B_I), \quad \emptyset \neq I \subseteq \{1, \ldots, t\}$$

where $B, B_i, B_I \subset S$ are G-graded ideals in S, well defined up to radical. We have

$$H^i_B(M) = H^i_Z(X, \tilde{M}), \text{ and } H^i_{B_I}(M) = H^i_{W_I}(X, \tilde{M}), \quad \emptyset \neq I \subseteq \{1, \ldots, t\}.$$

These local cohomology modules are G-graded.

Corollary 2.5. *Fix $i \geq 0$ and $\mathbf{d} \in G$. Suppose that, for all $\emptyset \neq I \subseteq \{1, 2, \ldots, t\}$, we have*

$$H^{i+\#I-1}_{B_I}(M)_{\mathbf{d}} = 0, \text{ then } H^i_B(M)_{\mathbf{d}} = 0.$$

Proof. This is an application of the above discussion and proposition 2.4. The terms of the spectral sequence contributing to $H^i_B(M)_{\mathbf{d}}$ are the $H^{i+\#I-1}_{B_I}(M)_{\mathbf{d}}$. □

Returning to the geometric situation:

Proposition 2.6. *Let $Z \subset X$ be a closed subset. Suppose that M is a sheaf of abelian groups on X with a resolution of the form*

$$0 \xrightarrow{\partial} C_d \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

There is a convergent spectral sequence

$$E_1^{-p,q} = H_Z^q(X, C_p) \Rightarrow H_Z^{q-p}(X, M)$$

Proof. Local cohomology is computed via Ext modules. Then apply Corollary 2.2. ☐

In our application, such sheaves and their resolutions will come from G-graded S-modules:

Corollary 2.7. *Let S be a G-graded R-algebra. Suppose $X = \operatorname{Spec}(S)$, M a G-graded S-module and suppose we have a resolution by G-graded S-modules:*

$$0 \xrightarrow{\partial} C_d \xrightarrow{\partial} \ldots \xrightarrow{\partial} C_0 \xrightarrow{\varepsilon} M \longrightarrow 0.$$

Let $B \subset S$ be a G-graded ideal. Fix $i \geq 0$ and $\mathbf{d} \in G$. Suppose that for all $p \geq 0$, we have $H_B^{i+p}(C_p)_{\mathbf{d}} = 0$, then $H_B^i(M)_{\mathbf{d}} = 0$.

Proof. Translating the sheaf language to module language, we see that the terms of the spectral sequence in the proposition contributing to $H_B^i(M)_{\mathbf{d}}$ are the $H_B^{i+p}(C_p)_{\mathbf{d}}$. ☐

Remark 2.8. If $\mathbf{P}_\Sigma$ is a smooth projective toric variety associated with the fan Σ, then Cox [3] has associated to $\mathbf{P}_\Sigma$ a multigraded algebra S and a monomial ideal $B \subset S$. In that case, we take $X = \operatorname{Spec}(S)$, with its T-action as above, and $Z = \mathbb{V}(B)$. This case will be discussed in Section 5.

3. Regularity regions and the degrees of syzygies

We assume that $S = K[x_1, ..., x_n]$ is a G-graded K-algebra, where K is a field, with $\deg x_i = \mathbf{a}_i \in G$. Let Q be the semigroup of G generated by the $\mathbf{a}_i$ for $1 \leq i \leq n$. We say S is positively multigraded by G, if G is torsion-free, $\deg x_i \neq \mathbf{0}$ for all i and Q has no non-zero invertible elements. This implies that each $S_{\mathbf{a}}$ is finite dimensional for each $\mathbf{a} \in Q$ ([14, theorem 8]). Any finitely generated G-graded S-module M has a finite free graded resolution each of whose terms is a finite direct sum of modules of the form

$$S(\mathbf{d}_1) \oplus \ldots \oplus S(\mathbf{d}_t)$$

([14, prop. 8.18]). If S is positively multigraded by G, then there is a well-defined notion of minimal free resolution, and any finitely generated G-graded module will have a unique up to isomorphism minimal resolution of this form. This is shown in the paragraph before proposition 8.18 in [14].

Recall the definition of regularity given in the introduction, Equation 1.2.

Definition 3.1. Let $B \subset S$ be a G-graded ideal in a G-graded ring $S = K[x_1, \cdots, x_n]$ and let M be a G-graded S-module. Define:

1. $\mathcal{Z}_B^i(M) = \{\mathbf{p} \in G : H_B^i(M)_{\mathbf{p}} = 0\}$.
2. $\operatorname{reg}_{B,\mathcal{C}}^i(M) = \{\mathbf{m} \in G : H_B^i(M)_{\mathbf{p}} = 0, \quad \forall \mathbf{p} \in \mathbf{m} + \mathbb{N}\mathcal{C}[1-i]\}$.
3. $\operatorname{reg}_{B,\mathcal{C}}(M) = \bigcap_{i \geq 0} \operatorname{reg}_B^i(M)$.

4. M is $\mathbf{m}$*-regular with respect to* B *and* $\mathcal{C}$ if $\mathbf{m} \in \mathrm{reg}_{B,\mathcal{C}}(M)$.

Remark 3.2. Note that $\mathrm{reg}^i_{B,\mathcal{C}}(M)$ is the largest set $A \subset G$ such that

$$A + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathcal{Z}^i_B(M).$$

Thus, $\mathrm{reg}_{B,\mathcal{C}}(M)$ is the largest set $A \subset G$ such that

$$A + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathcal{Z}^i_B(M), \quad \forall i \geq 0.$$

Our first aim is to estimate $\mathrm{reg}^i_{B,\mathcal{C}}(M)$ in terms of $\mathrm{reg}^i_{B,\mathcal{C}}(S)$ and the structure of a free graded resolution of M. The point of this is that the vanishing regions for $H^i_B(S)$ with respect to monomial ideals B are effectively computable. See the paper of Eisenbud, Mustata and Stillman [8] and also [16, Theorem 2.1] and [13, Proposition 3.2].

Definition 3.3. Suppose that $J = \{J_0, ..., J_s\}$ is a list of finite subsets $J_i \subset G$. A resolution of a G-graded S-module M of the form

$$0 \to \bigoplus_{\mathbf{d}_s \in J_s} S(-\mathbf{d}_s) \to \cdots \to \bigoplus_{\mathbf{d}_0 \in J_0} S(-\mathbf{d}_0) \to M \to 0$$

is called a resolution of type J.

Definition 3.4. Suppose that $J = \{J_0, ..., J_s\}$ is a list of finite subsets $J_i \subset G$, and suppose that we are given regions $\mathcal{V}^i_B(S) \subset \mathcal{Z}^i_B(S)$, for all $i \geq 0$. Define

$$\mathcal{D}^i_B(J) = \bigcap_{p=0}^{s} \bigcap_{\mathbf{d}_p \in J_p} (\mathbf{d}_p + \mathcal{V}^{i+p}_B(S)).$$

Remark 3.5. In the above definition, the notation does not reflect the choice of the sets $\mathcal{V}^i_B(S)$, which is implicit. We could take $\mathcal{V}^i_B(S) = \mathcal{Z}^i_B(S)$, for every i, but the reason for allowing greater flexibility in choosing the sets $\mathcal{V}^i_B(S)$ is that the complete vanishing region $\mathcal{Z}^i_B(S)$ might have a complicated form; or in any case a "nice" region, for instance, rational convex polyhedral subset $\mathcal{V}^i_B(S) \subset \mathcal{Z}^i_B(S)$ might be known or relevant. By the theorems of Eisenbud, Mustata and Stillman [8] such regions can often be found. In the lemmas that follow we let $\mathcal{V}^i_B(S) \subset \mathcal{Z}^i_B(S)$, $i \geq 0$ be any selection of regions and $\mathcal{D}^i_B(J)$ the corresponding sets.

Since

$$H^q_B(C_p)_{\mathbf{d}} = \bigoplus_{\mathbf{d}_p \in J_p} H^q_B(S)_{\mathbf{d}-\mathbf{d}_p},$$

it follows that

$$\mathcal{Z}^i_B(C_p) = \bigcap_{\mathbf{d}_p \in J_p} (\mathbf{d}_p + \mathcal{Z}^i_B(S)), \ \mathrm{reg}^i_B(C_p) = \bigcap_{\mathbf{d}_p \in J_p} (\mathbf{d}_p + \mathrm{reg}^i_B(S)). \tag{3.1}$$

Lemma 3.6. *Suppose that M has a resolution of type J. Then for each fixed $i \geq 0$ we have*

$$\mathcal{D}^i_B(J) \subseteq \mathcal{Z}^i_B(M).$$

Proof. For any fixed $p \geq 0$ it follows from Equation 3.1 that

$$\mathbf{d} \in \bigcap_{\mathbf{d}_p \in J_p} (\mathbf{d}_p + \mathcal{Z}^{i+p}_B(S)) \Rightarrow H^{i+p}_B(C_p)_{\mathbf{d}} = 0.$$

If $\mathbf{d} \in \mathcal{D}_B^i(J)$, this will hold for every $p \geq 0$, by the definition of the set $\mathcal{D}_B^i(J)$. If we assume that M has a resolution of type J, then Corollary 2.7 implies

$$H_B^i(M)_{\mathbf{d}} = 0$$

and therefore $\mathbf{d} \in \mathcal{Z}_B^i(M)$. □

Lemma 3.7. *Let $J = \{J_0, ..., J_s\}$ be a list of finite subsets $J_i \subset G$. Let $\mathbf{m} \in G$ and suppose that for a fixed $i \geq 0$,*

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathcal{D}_B^i(J).$$

Let M be a G-graded S-module with a resolution of type J. Then $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}^i(M)$. If this holds for every $i \geq 0$, then $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}(M)$.

Proof. The result follows by Lemma 3.6 and the definition of regularity 3.1. □

The next series of lemmas estimate the regularity of M in terms of the regularity of S.

Lemma 3.8. *Let $J = \{J_0, ..., J_s\}$ be a list of finite subsets $J_i \subset G$. Let $\mathbf{m} \in G$. Suppose that M is a G-graded S-module with a resolution of type J. If that for a fixed $i \geq 0$,*

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \bigcap_{p=0}^{s} \bigcap_{\mathbf{d}_p \in J_p} (\mathbf{d}_p + \operatorname{reg}_{B,\mathcal{C}}^{i+p}(S) + \mathbb{N}\mathcal{C}[1-i-p]),$$

then $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}^i(M)$. If this holds for every $i \geq 0$, then $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}(M)$.

Proof. By Remark 3.2

$$\operatorname{reg}_{B,\mathcal{C}}^{i+p}(S) + \mathbb{N}\mathcal{C}[1-i-p] \subseteq \mathcal{Z}_B^{i+p}(S),$$

so we can take $\mathcal{V}_B^{i+p}(S) = \operatorname{reg}_{B,\mathcal{C}}^{i+p}(S) + \mathbb{N}\mathcal{C}[1-i-p]$ in the definition of $\mathcal{D}_B^i(J)$, which shows that this is a special case of Lemma 3.7. □

Definition 3.9. Let $J = \{J_0, J_1, \ldots, J_s\}$ with $J_i \subseteq G$ any subset. For each $i \geq 0$, define the $\mathbb{N}\mathcal{C}$-modules

$$\operatorname{reg}_{B,\mathcal{C}}^i(J) = \{\mathbf{m} \in G : \mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \bigcap_{p=0}^{s} \bigcap_{\mathbf{d} \in J_p} \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p]\}$$

and

$$\operatorname{reg}_{B,\mathcal{C}}(J) = \bigcap_{i \geq 0} \operatorname{reg}_{B,\mathcal{C}}^i(J).$$

Lemma 3.10. *Suppose M is a G-graded S-module with a resolution of type J. Then for each $i \geq 0$*

$$\operatorname{reg}_{B,\mathcal{C}}^i(J) \subseteq \operatorname{reg}_{B,\mathcal{C}}^i(M), \text{ and } \operatorname{reg}_{B,\mathcal{C}}(J) \subseteq \operatorname{reg}_{B,\mathcal{C}}(M).$$

Proof. If $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}^i(J)$ then

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p].$$

for every $p \geq 0$ and $\mathbf{d} \in J_p$. But $\operatorname{reg}_{B,\mathcal{C}}(S) \subset \operatorname{reg}_{B,\mathcal{C}}^{i+p}(S)$, so $\mathbf{m}$ satisfies the hypotheses of Lemma 3.8, and the results follow. □

Lemma 3.11. *Let* $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_l\}$.

1. *If* $k \leq 0$, *then*

$$\mathbb{N}\mathcal{C}[k-1] = \bigcup_{j=1}^{l} (-\mathbf{c}_j + \mathbb{N}\mathcal{C}[k])$$

2. *If* $k \geq 1$, *then*

$$\mathbb{N}\mathcal{C}[k-1] \subseteq \bigcap_{j=1}^{l} (-\mathbf{c}_j + \mathbb{N}\mathcal{C}[k])$$

Proof. Both of these are rather clear. □

We have a simpler description of $\mathrm{reg}_{B,\mathcal{C}}(J)$:

Theorem 3.12. *Let* $\mathcal{C} = \{\mathbf{c}_1, \ldots, \mathbf{c}_l\}$.

$$\mathrm{reg}_{B,\mathcal{C}}(J) = X \cap Y$$

where

$$X = \bigcap_{\mathbf{d} \in J_0} (\mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[0])$$
$$Y = \bigcap_{p=1}^{s} \bigcap_{\mathbf{d} \in J_p} \bigcap_{j=1}^{l} (\mathbf{d} - \mathbf{c}_j + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p])$$

Proof. We will prove:

1.) $\mathrm{reg}_{B,\mathcal{C}}(J) = \mathrm{reg}^0_{B,\mathcal{C}}(J) \cap \mathrm{reg}^1_{B,\mathcal{C}}(J)$.
2.) $\mathrm{reg}^0_{B,\mathcal{C}}(J) = \bigcap_{j=1}^{l} \bigcap_{p=0}^{s} \bigcap_{\mathbf{d} \in J_p} (\mathbf{d} - \mathbf{c}_j + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p])$.
3.) $\mathrm{reg}^1_{B,\mathcal{C}}(J) = \bigcap_{p=0}^{s} \bigcap_{\mathbf{d} \in J_p} (\mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[-p])$.

Assuming these, we have

$$\mathrm{reg}_{B,\mathcal{C}}(J) = \bigcap_{p=0}^{s} \bigcap_{\mathbf{d} \in J_p} X_{p,\mathbf{d}} \cap Y_{p,\mathbf{d}},$$

where

$$X_{p,\mathbf{d}} = \mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[-p],$$
$$Y_{p,\mathbf{d}} = \bigcap_{j=1}^{l} (\mathbf{d} - \mathbf{c}_j + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p]).$$

But we will see in a moment that $X_{0,\mathbf{d}} \subseteq Y_{0,\mathbf{d}}$ and $Y_{p,\mathbf{d}} \subseteq X_{p,\mathbf{d}}$ if $p \geq 1$. Since $X = \bigcap_{\mathbf{d} \in J_0} X_{0,\mathbf{d}}$ and $Y = \bigcap_{p=1}^{s} \bigcap_{\mathbf{d} \in J_p} Y_{p,\mathbf{d}}$, this proves the proposition.

The inclusion $X_{0,\mathbf{d}} \subseteq Y_{0,\mathbf{d}}$ follows easily from Lemma 3.11 applied to $k = 1$.

Similarly $Y_{p,\mathbf{d}} \subseteq X_{p,\mathbf{d}}$ for $p \geq 1$ follows from the same lemma applied to $k = 1 - p \leq 0$.

Now we will prove the three claims:

1.) To show $\operatorname{reg}_{B,\mathcal{C}}(J) = \operatorname{reg}^0_{B,\mathcal{C}}(J) \cap \operatorname{reg}^1_{B,\mathcal{C}}(J)$, it will suffice to prove that $\operatorname{reg}^i_{B,\mathcal{C}}(J) \subseteq \operatorname{reg}^{i+1}_{B,\mathcal{C}}(J)$ for all $i \geq 1$.

$$\begin{aligned}\mathbf{m} \in \operatorname{reg}^i_{B,\mathcal{C}} \Leftrightarrow\ & \mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p], \\ & \forall p \geq 0, \quad \forall \mathbf{d} \in J_p. \\ \Rightarrow\ & \mathbf{m} + \mathbb{N}\mathcal{C}[1-i] - \mathbf{c}_j \subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p] - \mathbf{c}_j, \\ & \forall p \geq 0, \quad \forall \mathbf{d} \in J_p, \quad \forall j = 1, ..., l. \\ \Rightarrow\ & \mathbf{m} + \mathbb{N}\mathcal{C}[1-i] - \mathbf{c}_j \subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-(i+1)-p], \\ & \forall p \geq 0, \quad \forall \mathbf{d} \in J_p, \quad \forall j = 1, ..., l.\end{aligned}$$

The last implication holds because of Lemma 3.11, using that $i+p \geq 1$. Since $i \geq 1$ we can apply that lemma again to conclude:

$$\begin{aligned}\mathbf{m} + \mathbb{N}\mathcal{C}[1-(i+1)] &= \mathbf{m} + \bigcup_{j=1}^{l} (\mathbb{N}\mathcal{C}[1-i] - \mathbf{c}_j) \\ &= \bigcup_{j=1}^{l} (\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] - \mathbf{c}_j) \\ \text{(from the above)} &\subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-(i+1)-p], \\ & \quad \forall p \geq 0, \quad \forall \mathbf{d} \in J_p, \\ & \Rightarrow \mathbf{m} \in \operatorname{reg}^{i+1}_{B,\mathcal{C}}(J).\end{aligned}$$

2.) We will show $\operatorname{reg}^0_{B,\mathcal{C}}(J) = \bigcap_{j=1}^{l} \bigcap_{p=0}^{s} \bigcap_{\mathbf{d} \in J_p} (\mathbf{d} - \mathbf{c}_j + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p])$.

$$\begin{aligned}\mathbf{m} \in \operatorname{reg}^0_{B,\mathcal{C}}(J) \Leftrightarrow\ & \mathbf{m} + \mathbb{N}\mathcal{C}[1] \subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p], \\ & \forall p \geq 0, \quad \forall \mathbf{d} \in J_p. \\ \Leftrightarrow\ & \mathbf{m} + \mathbb{N}\mathcal{C} + \mathbf{c}_j \subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p] \\ & \forall p \geq 0, \quad \forall \mathbf{d} \in J_p, \quad \forall j = 1, ..., l. \\ \Leftrightarrow\ & \mathbf{m} + \mathbf{c}_j \in \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p], \\ & \forall p \geq 0, \quad \forall \mathbf{d} \in J_p, \quad \forall j = 1, ..., l.\end{aligned}$$

This last equivalence follows because $\mathbf{0} \in \mathbb{N}\mathcal{C}$ and each $\mathbb{N}\mathcal{C}[j]$ is an $\mathbb{N}\mathcal{C}$-module. This proves the claim.

3.)Finally, we will prove $\operatorname{reg}^1_{B,\mathcal{C}}(J) = \bigcap_{p=0}^{s} \bigcap_{\mathbf{d} \in J_p} (\mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[-p])$.

$$\begin{aligned}\mathbf{m} \in \operatorname{reg}^1_{B,\mathcal{C}}(J) \Leftrightarrow\ & \mathbf{m} + \mathbb{N}\mathcal{C} \subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[-p], \\ & \forall p \geq 0, \quad \forall \mathbf{d} \in J_p. \\ \Leftrightarrow\ & \mathbf{m} \in \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[-p], \\ & \forall p \geq 0, \quad \forall \mathbf{d} \in J_p, \quad \forall j = 1, ..., l.\end{aligned}$$

□

Suppose that a region $\mathcal{D} \subseteq G$ is given. We would like to define regions K_p such that if M has a resolution of type J with $J \subseteq K$, that is $J_i \subseteq K_i$ for all i, then $\mathcal{D} \subseteq \operatorname{reg}_{B,\mathcal{C}}(M)$.

Definition 3.13. For $\mathcal{D} \subseteq G$, we define $\operatorname{dreg}_{B,\mathcal{C}}(\mathcal{D}) = \{K_0, K_1, \dots\}$, where

$$K_p = \{\mathbf{d} \in G : \mathcal{D} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p], \quad \forall i \geq 0\}.$$

We also denote

$$K_{p,i} = \{\mathbf{d} \in G : \mathcal{D} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p]\},$$

and $\mathrm{dreg}_{B,\mathcal{C}}(\mathcal{D})_p = K_p = \bigcap_{i\geq 0} K_{p,i}$.

We drop B or $\mathcal{C}$ from the notation if it is clear.

Lemma 3.14. *Suppose that $\mathcal{D} \subseteq G$ is given and that M is a G-graded S-module of type J with $J \subseteq \mathrm{dreg}_{B,\mathcal{C}}(\mathcal{D})$. Then $\mathcal{D} \subseteq \mathrm{reg}_{B,\mathcal{C}}(M)$.*

Proof. By definition, if $\mathrm{dreg}_{B,\mathcal{C}}(\mathcal{D}) = \{K_0, K_1, \dots\}$, then

$$\mathcal{D} + \mathbb{N}\mathcal{C}[1-i] \subseteq \bigcap_{\mathbf{d}\in K_p} (\mathbf{d} + \mathrm{reg}(S) + \mathbb{N}\mathcal{C}[1-i-p]), \quad \forall p \geq 0.$$

Since $J \subseteq K$, by which we mean $J_p \subseteq K_p$ for all p,

$$\bigcap_{\mathbf{d}\in K_p} (\mathbf{d} + \mathrm{reg}(S) + \mathbb{N}\mathcal{C}[1-i-p]) \subseteq \bigcap_{\mathbf{d}\in J_p} (\mathbf{d} + \mathrm{reg}(S) + \mathbb{N}\mathcal{C}[1-i-p]).$$

Combining these with Lemma 3.8 gives the result. □

Lemma 3.15. *Let $\bar{\mathcal{D}}$ be the $\mathbb{N}\mathcal{C}$-module generated by $\mathcal{D}$. Then*

$$\mathrm{dreg}_{B,\mathcal{C}}(\mathcal{D}) = \mathrm{dreg}_{B,\mathcal{C}}(\bar{\mathcal{D}}).$$

Proof. It is true since $\mathbb{N}\mathcal{C}[i]$ is an $\mathbb{N}\mathcal{C}$-module for each i. □

We can give a simpler description of these regions:

Theorem 3.16. *Let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_l\}$. Then $\mathbf{d} \in \mathrm{dreg}_{B,\mathcal{C}}(\mathcal{D})_p$ if and only if*

$$\mathcal{D} \subseteq (\mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[-p]) \cap \bigcap_{j=1}^{l} (\mathbf{d} - \mathbf{c}_j + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p]).$$

Proof. By Definition 3.13, $\mathrm{dreg}_{B,\mathcal{C}}(\mathcal{D})_p = K_p = \bigcap_{i\geq 0} K_{p,i}$. If we can show $K_{p,i} \subseteq K_{p,i+1}$ for $i \geq 1$, then $K_p = K_{p,0} \cap K_{p,1}$. Namely,

$$\mathbf{d} \in K_{p,i} \Leftrightarrow \mathcal{D} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p].$$

When $i = 0$ this is equivalent to

$$\mathbf{d} \in K_{p,0} \Leftrightarrow \mathcal{D} + \mathbf{c}_j \subseteq \mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p].$$

for all $j = 1, ..., l$, since each $\mathbb{N}\mathcal{C}[k]$ is an $\mathbb{N}\mathcal{C}$-module. Or:

$$\mathbf{d} \in K_{p,0} \Leftrightarrow \mathcal{D} \subseteq \bigcap_{j=1}^{l} (\mathbf{d} - \mathbf{c}_j + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p]).$$

Similarly, when $i = 1$,

$$\mathbf{d} \in K_{p,1} \Leftrightarrow \mathcal{D} \subseteq \mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[-p].$$

Now we show $K_{p,i} \subseteq K_{p,i+1}$ for $i \geq 1$.

$$\begin{aligned}
\mathbf{d} \in K_{p,i} &\Leftrightarrow \mathcal{D} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p] \\
&\Rightarrow \mathcal{D} + \mathbb{N}\mathcal{C}[1-i] - \mathbf{c}_j \subseteq \mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-i-p] - \mathbf{c}_j, \\
&\quad \forall j = 1, ..., l. \\
&\Rightarrow \mathcal{D} + \mathbb{N}\mathcal{C}[1-i] - \mathbf{c}_j \subseteq \mathbf{d} + \mathrm{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-(i+1)-p], \\
&\quad \forall j = 1, ..., l.
\end{aligned}$$

The last implication holding because of Lemma 3.11 since $1-i-p \leq 0$. Since $i \geq 1$ we can apply that lemma again to conclude:

$$\begin{aligned}
\mathcal{D} + \mathbb{N}\mathcal{C}[1-(i+1)] &= \mathcal{D} + \bigcup_{j=1}^{l} (\mathbb{N}\mathcal{C}[1-i] - \mathbf{c}_j) \\
&\subseteq \bigcup_{j=1}^{l} (\mathcal{D} + \mathbb{N}\mathcal{C}[1-i] - \mathbf{c}_j) \\
\text{(from the above)} &\subseteq \mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-(i+1)-p].
\end{aligned}$$

By Definition 3.13, this implies that $\mathbf{d} \in K_{p,i+1}$. Therefore, we have $\mathbf{d} \in \operatorname{dreg}_{B,\mathcal{C}}(\mathcal{D})_p = K_p$ if and only if

$$\mathcal{D} \subseteq (\mathbf{d} + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[-p]) \cap \bigcap_{j=1}^{l} (\mathbf{d} - \mathbf{c}_j + \operatorname{reg}_{B,\mathcal{C}}(S) + \mathbb{N}\mathcal{C}[1-p]).$$

□

The above results imply that if the region $\mathcal{D}$ is finitely generated as an $\mathbb{N}\mathcal{C}$-module the conditions defining $\operatorname{dreg}(\mathcal{D})$ are finite in number - a priori, there are only finitely many possible sets K_p to be considered in the free resolutions of any module.

The following is easily verified:

Corollary 3.17. *The mappings*

$$\begin{pmatrix} \text{collections} \\ K = \{K_0, K_1, \ldots, K_s\} \\ K_i \subseteq G \end{pmatrix} \begin{array}{c} \text{reg} \\ \rightarrow \\ \leftarrow \\ \text{dreg} \end{array} \begin{pmatrix} \mathbb{N}\mathcal{C} - \text{modules} \\ \mathcal{D} \subseteq G \end{pmatrix}$$

satisfy

1. *If $K \subseteq K'$ denotes $K_i \subseteq K_i'$ for all $i \geq 0$, then* $\operatorname{reg}_{B,\mathcal{C}}(K) \supseteq \operatorname{reg}_{B,\mathcal{C}}(K')$.
2. *If $\mathcal{D} \subseteq \mathcal{D}'$, then* $\operatorname{dreg}_{B,\mathcal{C}}(\mathcal{D}) \supseteq \operatorname{dreg}_{B,\mathcal{C}}(\mathcal{D}')$.
3. $K \subseteq \operatorname{dreg}_{B,\mathcal{C}}(\operatorname{reg}_{B,\mathcal{C}}(K))$.
4. $\mathcal{D} \subseteq \operatorname{reg}_{B,\mathcal{C}}(\operatorname{dreg}_{B,\mathcal{C}}(\mathcal{D}))$.

4. $\mathbf{v}$-gradings

Let $S = K[x_1, \ldots, x_n]$ be a G-graded algebra where $G = \mathbb{Z}^r$.

Definition 4.1. A vector $\mathbf{v} \in \mathbb{Z}^r$ is called a *coarsening vector* for S if

$$\deg_{\mathbf{v}}(x_i) = \mathbf{v} \cdot \deg(x_i) \geq 0, \quad \forall 1 \leq i \leq n.$$

It is called a *positive coarsening vector* if $\deg_{\mathbf{v}}(x_i) > 0$ for all i.

The ring S and all G-graded S-modules M inherit $\mathbb{Z}$-gradings from $\mathbf{v}$. We denote these $\mathbb{Z}$-graded objects by $S_{\mathbf{v}}$ and $M_{\mathbf{v}}$, where

$$(M_{\mathbf{v}})_m = \bigoplus_{\mathbf{d} \in G, \mathbf{v} \cdot \mathbf{d} = m} M_{\mathbf{d}}.$$

We can regard the variables x_i with $\mathbf{v} \cdot \deg(x_i) = 0$ as constants. That is we can write $S_{\mathbf{v}} = R[x_j]$ where x_j with $\mathbf{v} \cdot \deg(x_i) > 0$, and $R = K[x_i]$ with $\mathbf{v} \cdot \deg(x_i) = 0$. Then $S_{\mathbf{v}}$ is the homogeneous coordinate ring of the weighted projective space over

R in the coordinates x_j. If the vector $\mathbf{v}$ is understood, we will simply write S and M, and the notation M_m for $m \in \mathbb{Z}$ will refer to this $\mathbf{v}$-grading.

Assume $\mathbf{v}$ is a positive coarsening vector. Let $c_{\mathbf{v}} = \operatorname{lcm}\{\deg_{\mathbf{v}}(x_i) : 1 \leq i \leq n\}$. Recall that $\mathbb{N}c_{\mathbf{v}}$ is the Kaehler cone for $S_{\mathbf{v}}$, for details see [13, Example 2.1].

Definition 4.2. If M is a finitely generated G-graded S-module and $\mathbf{v}$ is a positive coarsening vector for S, then

$$\operatorname{reg}_{\mathbf{v}}(M) = \{p \in \mathbb{Z} : H^i_{\mathfrak{m}}(M)_q = 0, \quad \forall q \in p + \mathbb{N}c_{\mathbf{v}}[1-i], \mathfrak{m} = (x_1, \ldots, x_n)\}.$$

This is $\mathfrak{m}$-regularity with respect to $\mathcal{C} = \{c_{\mathbf{v}}\}$ defined by [13] for the $\mathbf{v}$-graded module M.

Here we will quote Theorem 4.3 and Theorem 4.4 from [19].

Theorem 4.3. *Let $M \neq 0$ be a finitely generated G-graded module. Let $\mathbf{v} \in G$ be a positive coarsening vector for S. Then there exists a $p \in \operatorname{reg}_{\mathbf{v}}(M)$ such that $q \geq p$ implies that $q \in \operatorname{reg}_{\mathbf{v}}(M)$. The least such p with this property is called* reg-num$_{\mathbf{v}}(M)$. *If $M = 0$, then we set* reg-num$_{\mathbf{v}}(M) = -\infty$.

Theorem 4.4. *Let $S, M, \mathbf{v}$ be as in the previous theorem. Let $C_\bullet$ be a minimal G-graded free resolution of M. Let*

$$s_{\mathbf{v}} = \max\{nc_{\mathbf{v}} - \sum \deg_{\mathbf{v}}(x_i), c_{\mathbf{v}}\}.$$

Then the degrees $\mathbf{d} \in G$ of the module C_i satisfy

$$\deg_{\mathbf{v}}(\mathbf{d}) \leq \text{reg-num}_{\mathbf{v}}(M) + is_{\mathbf{v}} + c_{\mathbf{v}} - 1.$$

Definition 4.5. Let $\mathbf{v}_1, \ldots, \mathbf{v}_t$ be positive coarsening vectors for S. Let $b_1, \ldots, b_t$ be integers. Let M be a finitely generated G-graded S-module. We say that M is *b_*-regular with respect to $\mathbf{v}_*$* if

$$b_j \in \operatorname{reg}_{\mathbf{v}_j}(M), \quad \forall 1 \leq j \leq t.$$

Corollary 4.6. *Suppose that M is b_*-regular with respect to $\mathbf{v}_*$, where each $b_j \geq$* reg-num$_{\mathbf{v}_j}(M)$. *Then the degrees $\mathbf{d}$ of the i-th syzygy module C_i in a minimal G-graded resolution of M belong to the convex polyhedral region*

$$K_i(\mathbf{v}_*, b_*) = \{\mathbf{d} \in G : \deg_{\mathbf{v}_j}(\mathbf{d}) \leq b_j + is_{\mathbf{v}_j} + c_{\mathbf{v}_j} - 1, \quad \forall 1 \leq j \leq t\}.$$

Remark 4.7. Let $Q \subseteq G$ be the sub-semigroup generated by the degrees $Q = \mathbb{N}\{\deg(x_i) : 1 \leq i \leq n\}$. Then the support of M is contained in a union of finitely many translates $\mathbf{b}_k + Q$. The supports of the syzygy module C_i in a minimal G-graded resolution of M will be contained in the same union of these translates, so we can actually say that under the hypotheses of Theorem 4.4, the degrees of C_i are in the finite set

$$(\bigcup_{j=1}^{k} (\mathbf{b}_j + Q)) \bigcap K_i(b_*, \mathbf{v}_*), \quad \text{for some } k.$$

Theorem 4.8. *Suppose that M is a finitely generated G-graded S-module, $\mathbf{v}_1, \ldots, \mathbf{v}_t \in \mathbb{Z}^r$ positive coarsening vectors for S, and $b_1, \ldots, b_t \in \mathbb{Z}$ integers such that $b_j \geq$* reg-num$_{\mathbf{v}_j}(M)$ *for $1 \leq j \leq t$. Suppose that M is b_*-regular with respect to $\mathbf{v}_*$. Let $J = \{J_0, J_1, ..., J_s\}$ where*

$$J_p = (\bigcup_{j=1}^{k} (\mathbf{b}_j + \mathbb{N}\mathcal{C})) \bigcap K_p(\mathbf{v}_*, b_*).$$

Then,

1. *M has a resolution of type J', with $J' = \{J'_0, \dots, J'_s\}$ and $J'_i \subseteq J_i$.*
2. $\operatorname{reg}(J) \subseteq \operatorname{reg}(J') \subseteq \operatorname{reg}_{B,\mathcal{C}}(M)$.

Proof. The first claim is a restatement of Remark 4.7. The second claim follows from the first claim and by Lemma 3.10. □

The above gives a lower bound on a regularity region for M in terms of a b_*-regularity region with respect to a collection of coarsening vectors. We now give a set of conditions, based on vanishing of $H^i_B(M)_{\mathbf{d}}$ for $\mathbf{d}$ in half planes, that imply $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}(M)$.

Definition 4.9. Let $\mathbf{v} \in \mathbb{R}^r$, $b \in \mathbb{R}$, define the half-planes and the hyperplane

$$P^+_{\mathbf{v},b} = \{\mathbf{x} \in \mathbb{R}^r : \mathbf{v} \cdot \mathbf{x} \geq b\}, \quad P^-_{\mathbf{v},b} = -P^+_{\mathbf{v},b},$$
$$L_{\mathbf{v},b} = P^-_{\mathbf{v},b} \cap P^+_{\mathbf{v},b} = \{\mathbf{x} \in \mathbb{R}^r : \mathbf{v} \cdot \mathbf{x} = b\}.$$

Lemma 4.10. *Let $\mathcal{C} = \{\mathbf{c}_1, \dots, \mathbf{c}_l\}$. Let $\mathbf{v} \in \mathbb{R}^r$. Suppose that*

$$\mathbb{N}\mathcal{C} \subseteq P^+_{\mathbf{v},0}, \text{ that is } \mathbf{v} \cdot \mathbf{c}_j \geq 0 \text{ for all } j = 1, \dots, l.$$

Note that such a vector $\mathbf{v}$ always exists under the assumption $\mathcal{C} \subset \mathcal{K}^{\mathrm{sat}}$ since the latter is a subsemigroup of $\mathbb{N}\mathcal{A}$ and we are assuming that $\operatorname{pos}(\overline{A})$ is a pointed cone (see the introduction). If we define $\min(\mathbf{v}, \mathbf{c}) = \min(\mathbf{v} \cdot \mathbf{c}_j)^l_{j=1}$, and $\max(\mathbf{v}, \mathbf{c}) = \max(\mathbf{v} \cdot \mathbf{c}_j)^l_{j=1}$. Then $\mathbb{N}\mathcal{C}[k] \subseteq P^+_{\mathbf{v},b}$, where $b = k \min(\mathbf{v}, \mathbf{c})$ if $k \geq 0$, and $b = k \max(\mathbf{v}, \mathbf{c})$ if $k < 0$.

Proof. For $k = 0$ it is true by assumption. Observe that

$$\mathbb{N}\mathcal{C}[k+1] = \bigcup_{j=1}^{l} \mathbf{c}_j + \mathbb{N}\mathcal{C}[k], \text{ if } k \geq 0,$$
$$\mathbb{N}\mathcal{C}[k-1] = \bigcup_{j=1}^{l} -\mathbf{c}_j + \mathbb{N}\mathcal{C}[k], \text{ if } k \leq 0.$$

We will prove the result by performing induction on k. Assume that $k \geq 0$. If $\mathbf{d} \in \mathbb{N}\mathcal{C}[k+1]$, then $\mathbf{d} = \mathbf{c}_j + \mathbf{e}$ for some $\mathbf{e} \in \mathbb{N}\mathcal{C}[k]$. By induction hypothesis, we have

$$\mathbf{v} \cdot \mathbf{d} = \mathbf{v} \cdot (\mathbf{c}_j + \mathbf{e}) \geq \mathbf{v} \cdot \mathbf{c}_j + k \min(\mathbf{v}, \mathbf{c}) \geq (k+1) \min(\mathbf{v}, \mathbf{c}).$$

By Definition 4.9, we have that $\mathbb{N}\mathcal{C}[k] \subseteq P^+_{\mathbf{v},b}$ where $b = k \min(\mathbf{v}, \mathbf{c})$.

Now, assume that $k \leq 0$. If $\mathbf{d} \in \mathbb{N}\mathcal{C}[k-1]$, then $\mathbf{d} = -\mathbf{c}_j + \mathbf{e}$ for some $\mathbf{e} \in \mathbb{N}\mathcal{C}[k]$. By induction hypothesis, we have

$$\mathbf{v} \cdot \mathbf{d} = \mathbf{v} \cdot (-\mathbf{c}_j + \mathbf{e}) \geq \mathbf{v} \cdot (-\mathbf{c}_j) + k \max(\mathbf{v}, \mathbf{c}) \geq (k-1) \max(\mathbf{v}, \mathbf{c}).$$

By Definition 4.9, we have that $\mathbb{N}\mathcal{C}[k] \subseteq P^+_{\mathbf{v},b}$ where $b = k \max(\mathbf{v}, \mathbf{c})$. □

Theorem 4.11. *Let $\mathbf{v} \in G$ be a coarsening vector for S such that $\mathbf{v} \cdot \mathbf{c}_j \geq 0$ for all $j = 1, \dots, l$. Let $k \in \mathbb{Z}$. Let M be a G-graded S-module. Fix $\mathbf{m} \in G$, and let $B \subseteq G$ be a G-graded ideal. Suppose that*

$$H^i_B(M)_{\mathbf{d}} = 0, \quad \forall \mathbf{d}, \text{ such that } \deg_{\mathbf{v}}(\mathbf{d}) \geq \deg_{\mathbf{v}}(\mathbf{m}) + (k-i) \cdot m(k,i),$$

where

$$m(k,i) = \max(\mathbf{v}, \mathbf{c}), \text{ if } k - i < 0, \text{ and } m(k,i) = \min(\mathbf{v}, \mathbf{c}), \text{ if } k - i \geq 0.$$

Then

$$\mathbf{m} + \mathbb{N}\mathcal{C}[k-i] \subseteq \mathcal{Z}^i_B(M).$$

In particular, if $k = 1$ and this holds for all $i \geq 0$, this says that $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}(M)$.

Proof. The condition states that

$$H^i_B(M)_{\mathbf{d}} = 0, \quad \forall \mathbf{d} - \mathbf{m} \in P^+_{\mathbf{v},b}, \text{ where } b = (k-i) \cdot m(k,i).$$

By Lemma 4.10, $\mathbb{N}\mathcal{C}[k-i] \subseteq P^+_{\mathbf{v},b}$. Therefore

$$H^i_B(M)_{\mathbf{d}} = 0, \quad \forall \mathbf{d} - \mathbf{m} \in \mathbb{N}\mathcal{C}[k-i],$$

that is

$$\mathbf{m} + \mathbb{N}\mathcal{C}[k-i] \subseteq \mathcal{Z}^i_{B,\mathcal{C}}(M).$$

By 2 and 3 in Definition 3.9, when $k = 1$ and all $i \geq 0$, we have $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}(M)$. $\square$

5. Regularity with respect to a family of ideals

Let (S, B) be a G-graded R-algebra with a monomial ideal. In this section we consider the case where there is a decomposition

$$Z = \mathbb{V}(B) = \bigcup_{i=1}^{t} \mathbb{V}(B_i) = \bigcup_{i=1}^{t} Z_i$$

$$W_I = \bigcap_{i \in I} Z_i = \mathbb{V}(B_I), \quad \emptyset \neq I \subseteq [t] = \{1, \ldots, t\}$$

If (S, B) is the homogeneous coordinate ring and irrelevant ideal of a toric variety $\mathbf{P}_\Sigma$ (see [3]), then there is a canonical decomposition of this type, due to Batryev (see [4, section 10]) in which the ideals B_I are defined by linear forms. Here is the definition: The cones of the fan Σ are spanned by integral vectors in a vector space $N_{\mathbb{R}}$. A collection of edge generators $\{\mathbf{n}_{i_1}, ..., \mathbf{n}_{i_r}\}$ is called primitive if they do not all lie in a cone of Σ but every subset of it does. Then, we have a decomposition as above with

$$B_i = \langle x_{i_1}, ..., x_{i_r} \rangle, \ B_I = \sum_{i \in I} B_i$$

where the index i runs over all the primitive sets.

Lemma 5.1. *For each $i \geq 0$*

$$\bigcap_{p \geq 0} \ \bigcap_{\substack{I \subseteq [t] \\ \#I = p+1}} \mathcal{Z}^{i+p}_{B_I}(M) \subseteq \mathcal{Z}^i_B(M).$$

Proof. Let $\mathbf{d} \in \bigcap_{p \geq 0} \bigcap_{\substack{I \subseteq [t] \\ \#I = p+1}} \mathcal{Z}^{i+p}_{B_I}(M)$, then $\mathbf{d} \in G$ and $H^{i+p}_{B_I}(M)_{\mathbf{d}} = 0$ for all $p \geq 0$ and $I \subseteq [t]$ with $\#I = p+1$ by Definition 3.9. By Corollary 2.5, this implies that $H^i_B(M)_{\mathbf{d}} = 0$, that is, $\mathbf{d} \in \mathcal{Z}^i_B(M)$. $\square$

Lemma 5.2. *Fix $i \geq 0$. If*

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \bigcap_{p \geq 0} \ \bigcap_{\substack{I \subseteq [t] \\ \#I = p+1}} \mathcal{Z}^{i+p}_{B_I}(M),$$

then $\mathbf{m} \in \operatorname{reg}^i_{B,\mathcal{C}}(M)$. If this is true for all i, then $\mathbf{m} \in \operatorname{reg}_{B,\mathcal{C}}(M)$.

Proof. By Lemma 5.1, this implies

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathcal{Z}^i_B(M).$$

By Definition 3.9, $\mathbf{m} \in \mathrm{reg}^i_{B,\mathcal{C}}(M)$; and $\mathbf{m} \in \mathrm{reg}_{B,\mathcal{C}}(M)$ if this is true for all i. $\square$

Definition 5.3. A G-graded S-module M is $\mathbf{m}$-regular in dimension i with respect to the collection $\{B_I\}$ and $\mathcal{C}$ if for all

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathcal{Z}^{i+\#I-1}_{B_I}(M), \text{ for all } \emptyset \neq I \subseteq [t].$$

We denote this by $\mathbf{m} \in \mathrm{reg}^i_{B_*,\mathcal{C}}(M)$, and define

$$\mathrm{reg}_{B_*,\mathcal{C}}(M) = \bigcap_{i\geq 0} \mathrm{reg}^i_{B_*,\mathcal{C}}(M).$$

Proposition 5.4. *Suppose that B_i is a family of monomial ideals of S giving a decomposition of a monomial ideal B as indicated above. Then*

$$\mathrm{reg}_{B_*,\mathcal{C}}(M) \subseteq \mathrm{reg}_{B,\mathcal{C}}(M).$$

Proof. If $\mathbf{m} \in \mathrm{reg}_{B_*,\mathcal{C}}(M)$, then by Definition 5.3,

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \bigcap_{p\geq 0} \bigcap_{\substack{I\subseteq[t] \\ \#I=p+1}} \mathcal{Z}^{i+p}_{B_I}(M), \quad \forall i \geq 0.$$

By Lemma 5.2, we have that $\mathbf{m} \in \mathrm{reg}^i_{B,\mathcal{C}}(M)$ for all $i \geq 0$. Therefore, $\mathbf{m} \in \bigcap_{i\geq 0} \mathrm{reg}^i_{B,\mathcal{C}}(M) = \mathrm{reg}_{B,\mathcal{C}}(M)$. $\square$

Assume $G = \mathbb{Z}^r$. Let us consider the case of a decomposition of $Z = \mathbb{V}(B)$. When discussing vanishing conditions for $H^i_{B_I}(M)$ it is natural to consider coarsening vectors that assign $\mathbf{0}$ to some of the variables. This is justified, that is, that one can consider coarsenings that assign zero to some of the variables, will be shown in the Section 6 (see especially Remark 6.6). In fact for each subset I one could consider a finite set of such coarsenings. We will do the simplest case of assigning only one coarsening vector to each $\emptyset \neq I$.

Definition 5.5. Let S be a $G = \mathbb{Z}^r$-graded R-algebra. Let $B_i,\ i \in [t]$ be a family of monomial ideals in S. For each $\emptyset \neq I \subseteq [t]$, let $\mathbf{v}_I \in G$ be a coarsening vector, not necessarily strict. Let M be a G-graded S-module. We say that $\mathbf{m} \in \mathrm{reg}^i_{B_*,\mathbf{v}_*,\mathcal{C}}$ if for all $\emptyset \neq I \subseteq [t]$,

$$H^i_{B_I}(M)_{\mathbf{d}} = 0 \text{ for all } \deg_{\mathbf{v}_I}(\mathbf{d}) \geq \deg_{\mathbf{v}_I}(\mathbf{m}) + (1-i)m(1,i)$$

where

$$m(1,i) = \max(\mathbf{v}_I, \mathbf{c}), \text{ if } 1-i<0, \text{ and } m(1,i) = \min(\mathbf{v}_I, \mathbf{c}), \text{ if } 1-i \geq 0.$$

Define

$$\mathrm{reg}_{B_*,\mathbf{v}_*,\mathcal{C}}(M) = \bigcap_{i\geq 0} \mathrm{reg}^i_{B_*,\mathbf{v}_*,\mathcal{C}}(M).$$

Proposition 5.6. *Notations as in Definition 5.5, if $\mathbf{v}_I \cdot \mathbf{c}_j \geq 0$ for all $\emptyset \neq I \subseteq [t]$ and all $\mathbf{c}_j \in \mathcal{C}$, then*

$$\mathrm{reg}_{B_*,\mathbf{v}_*,\mathcal{C}}(M) \subseteq \mathrm{reg}_{B_*,\mathcal{C}}(M).$$

If the ideals B_i arise from a decomposition of $\mathbb{V}(B)$ as explained at the beginning of this section, then

$$\mathrm{reg}_{B_*,\mathbf{v}_*,\mathcal{C}}(M) \subseteq \mathrm{reg}_{B,\mathcal{C}}(M).$$

Proof. If $\mathbf{m} \in \mathrm{reg}_{B_*,\mathbf{v}_*,\mathcal{C}}(M)$ then for all $\emptyset \neq I \subseteq [t]$, the conditions of Theorem 4.11 hold with $B = B_I$, $\mathbf{v} = \mathbf{v}_I$ and $k = 1$. From the result of that Theorem, we conclude that

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathcal{Z}^i_{B_I}(M), \text{ for all } i \geq 0.$$

In particular,

$$\mathbf{m} + \mathbb{N}\mathcal{C}[(2-\#I)-i] \subseteq \mathcal{Z}^{i+\#I-1}_{B_I}(M), \text{ for all } i \geq 0.$$

But $(2-\#I)-i \leq 1-i$, thus

$$\mathbf{m} + \mathbb{N}\mathcal{C}[1-i] \subseteq \mathbf{m} + \mathbb{N}\mathcal{C}[(2-\#I)-i] \subseteq \mathcal{Z}^{i+\#I-1}_{B_I}(M), \text{ for all } i \geq 0.$$

By Definition 5.3, $\mathbf{m} \in \bigcap_{i\geq 0} \mathrm{reg}^i_{B_*,\mathcal{C}}(M) = \mathrm{reg}_{B_*,\mathcal{C}}(M)$.

If we assume that the $B_* = \{B_i\}$ comes from a decomposition of a monomial ideal B, then $\mathbf{m} \in \mathrm{reg}_{B_*,\mathcal{C}}(M) \subseteq \mathrm{reg}_{B,\mathcal{C}}(M)$ by Proposition 5.4. □

Often it is the case that if $\mathbf{m} \in \mathrm{reg}_{B_*,\mathbf{v}_*,\mathcal{C}}(M)$, we can find a resolution for M with syzygies in a particular region depending on $\mathbf{m}$. The point of considering a family of ideals is that imposing regularity conditions with respect to each ideal B_I gives a family of constraints, indexed by I, on the degrees of syzygies of a minimal resolution of a module. To apply the methods of Section 4 here, it is necessary to assume that the coarsenings $\mathbf{v}_I$ are orthogonal in the following sense:

Assume that each B_I is generated by a subset of generators x_j of $S = K[x_1, ..., x_n]$. Then we must postulate that $\deg_{\mathbf{v}_I}(x_j) > 0$ for every variable $x_j \in B_I$, and that $\deg_{\mathbf{v}_I}(x_j) = 0$ for every variable $x_j \notin B_I$. Such a vector may not exist, in general, but they do exist for fans of the form $\Sigma = \Sigma_1 \times ... \times \Sigma_r$.

For instance, if we assume that the variables in B_i are orthogonal to those of B_j for $i \neq j$ in the sense that we can choose coarsenings $\mathbf{v}_i$ for the variables $x_k \in B_i$ with $\deg_{\mathbf{v}_i}(x_k) > 0$ for every variable $x_k \in B_i$, and $\deg_{\mathbf{v}_I}(x_k) = 0$ for every variable $x_k \notin B_j$, we can define $\mathbf{v}_I = \sum_{i\in I} \mathbf{v}_i$ since $B_I = \sum_{i\in I} B_i$.

Proposition 5.7. *Let B_i, $i \in [t]$ be the set of ideals arising from the Batryev decomposition of a toric variety, as in the introduction of this section. Recall that each ideal B_I is generated by a subset of generators $\{x_1, ..., x_n\}$ of $S = K[x_1, ..., x_n]$. Suppose that coarsening vectors are chosen $\mathbf{v}_I$ for $\emptyset \neq I \subseteq [t]$ with the property that $\deg_{\mathbf{v}_I}(x_j) > 0$ for every variable $x_j \in B_I$ and that $\deg_{\mathbf{v}_I}(x_j) = 0$ for every variable $x_j \notin B_I$.*

Let M be a $G = \mathbb{Z}^r$-graded S-module and $\mathbf{m} \in \mathrm{reg}_{B_*,\mathbf{v}_*,\mathcal{C}}(M)$. Suppose for each I integers b_I are given with $b_I \geq \text{reg-num}_{\mathbf{v}_\mathbf{I}}(M)$.

Assume that for every I, and $i = 0, ..., n$,

$$\deg_{\mathbf{v}_I}(\mathbf{m}) \leq b_I + (1-i)(c_{\mathbf{v}_I} - m(1,i)).$$

If we let $\mathbf{v}_*$ be the collection $\{\mathbf{v}_I\}$ and b_* be the collection of integers $\{b_I\}$, then M has a resolution of type J' where J' is defined as in Theorem 4.8.

Proof. The condition that $\mathbf{m} \in \mathrm{reg}_{B_*,\mathbf{v}_*,\mathcal{C}}(M)$ and the given bounds on $\deg_{\mathbf{v}_I}(\mathbf{m})$ imply that for all $\mathbf{d}$ with

$$\deg_{\mathbf{v}_I}(\mathbf{d}) \geq b_I + (1-i)c_{\mathbf{v}_I}.$$

Therefore, we have $H^i_{B_I}(M)_\mathbf{d} = 0$. Since $b_I \geq \text{reg-num}_{\mathbf{v}_\mathbf{I}}(M)$, which holds for all I, this is equivalent to the condition that M is b_*-regular with respect to $\mathbf{v}_*$. Thus we may apply Theorem 4.8 to get the result. □

6. REGULARITY FOR G-GRADED R-ALGEBRAS

In this section we show how the theory developed so far can be extended from a G-graded polynomial ring $S = K[x_1, ..., x_n]$ where K is a field to $S = R[x_1, ..., x_n]$ where R is a commutative Noetherian ring. If $S = R[x_1, ..., x_n]$, a finitely generated G-graded S-module may not have a finite free graded resolution of the type we have been considering. In our applications this is no restriction: we always start with a module over $S = K[x_1, ..., x_n]$ which a priori has such free G-graded finite resolution, and we seek bounds on the degrees of generators of syzygies by consideration of vector gradings that possibly assign degree zero to some of the variables (see Remark 6.6).

Note that all the results of section 3 are true as stated, with the ground field K replaced by any commutative ring R. First we define regularity as in the introduction, but for $S = R[x_1, ..., x_n]$. Then the results of section 3 follow formally from the spectral sequence arguments of section 2: we are postulating the free resolutions for our modules.

The results of section 4 depend on the paper [19], whose main results depend on Maclagan and Smith's paper. We will argue that the main results, especially [13, Theorem 4.7, Theorem 5.4] are true as stated in this more general setting.

This is done in two stages:

a. The theory works for R, a local ring with infinite residue field.
b. If the theory works for all local rings with infinite residue field, then it works for all commutative Noetherian rings.

By "theory works" we mean that the results stated in [13] and [19] are valid for the ring $S = R[x_1, ..., x_n]$ where R is the type of ring under discussion. In all cases, $B \subset S$ is a monomial ideal, possibly subject to further restrictions as needed in those cited works. The main results of those works are of the following sort:

1. Knowing that $H^i_B(M)_{\mathbf{d}} = 0$ for $\mathbf{d}$ in a certain region of G, deduce that $H^i_B(M)_{\mathbf{d}} = 0$ for $\mathbf{d}$ in a larger region.
2. From the vanishing $H^i_B(M)_{\mathbf{d}} = 0$ for $\mathbf{d}$ in certain regions of G depending on i, deduce that the degrees of the generators of S-module M may be found in a region of G. More generally the degrees of the pth syzygies are in a certain region.

Let us first show how assuming we have verified step a. above we may conclude step b. The idea, due to Ooishi [17], is to consider the localizations $R \to R_{\mathfrak{p}} \to R_{\mathfrak{p}}(T)$ where $\mathfrak{p}$ ranges over $\mathrm{Spec}(R)$ and T is a variable. The first arrow is flat and the second arrow is faithfully flat. Thus, if $P \subset Q$ are R-modules, then $P = Q$ if and only if $P_{\mathfrak{p}}(T) = Q_{\mathfrak{p}}(T)$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. Applied to $P = 0$, $Q = H^i_B(M)_{\mathbf{d}}$ we see that $H^i_B(M)_{\mathbf{d}} = 0$ if and only if $H^i_B(M)_{\mathbf{d}} \otimes_R R_{\mathfrak{p}}(T) = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R)$. But the formation of local cohomology commutes with flat base-change: $H^i_B(M)_{\mathbf{d}} \otimes_R R_{\mathfrak{p}}(T) = H^i_{B_{\mathfrak{p}}(T)}(M_{\mathfrak{p}}(T))_{\mathbf{d}}$. Thus problem 1. above is reduced to a set of similar problems over each $R_{\mathfrak{p}}(T)[x_1, ..., x_n]$, and $R_{\mathfrak{p}}(T)$ is a local ring with infinite residue field.

As for problem 2. the issue is to know whether the R-submodule P of some $Q = M_{\mathbf{d}}$ generated by elements from $S_{\mathbf{d}-\mathbf{e}} M_{\mathbf{e}}$ for all the $\mathbf{e}$ in some given region of G, is all of $Q = M_{\mathbf{d}}$. We check this by all localizations $R_{\mathfrak{p}}(T)$. We are thus reduced to handling problem 1 and 2 when R is a local ring with infinite residue field.

Therefore we must verify a. above. The one serious place where the hypothesis "K is an infinite field" is used is to assert the existence of generic forms $g \in S_{\mathbf{d}}$ with good properties, which are then utilized in induction arguments. Actually Maclagan and Smith only assume that K is a field and base extend to an infinite field. Ooishi also observed that the existence of these forms can be shown if the ground ring is a local ring with infinite residue field.

First we recall some definitions.

Definition 6.1. A module M is *B-torsion* if $M = H^0_B(M)$. If M is B-torsion, then $H^i_B(M) = 0$ for all $i > 0$.

Definition 6.2. For an element $g \in S$, we set

$$(0 :_M g) = \{f \in M : gf = 0\}.$$

This submodule is zero when g is a non-zero-divisor on M. We say that $g \in S$ is *almost a non-zero-divisor on M* if $(0 :_M g)$ is a B-torsion module.

Definition 6.3. We say that a commutative ring R is a *u-ring* provided R has the property that an ideal contained in a finite union of ideals must be contained in one of those ideals; and a *um-ring* is a ring R with the property that an R-module which is equal to a finite union of submodules must be equal to one of them.

Theorem 6.4. *R is a um-ring if and only if the residue field $R/\mathfrak{p}$ is infinite for each maximal ideal $\mathfrak{p}$ of R, and R is a u-ring if and only if for each maximal ideal $\mathfrak{p}$ of R either the residue field $R/\mathfrak{p}$ is infinite or the quotient ring $R_{\mathfrak{p}}$ is a Bezout ring.*

Proof. For details see [Theorem 2.2, Theorem 2.3 and Theorem 2.6 [18]]. $\square$

The existence of forms with good properties for induction is the following (see [13, Prop. 3.1]).

Proposition 6.5. *Let $S = R[x_1, ..., x_n]$ be a G-graded polynomial ring where R is a Noetherian local ring with an infinite residue field. Let $B \subset S$ be a nonzero monomial ideal attached to a maximal cell Γ as explained in* [13, section 1] *(see also section (1) of this paper). Let M be a finitely generated G-graded S-module. If $\mathbf{p} \in \mathcal{K}$, with $\mathcal{K}$ the Kaehler cone, and $g \in S_{\mathbf{p}}$ is a sufficiently general form, then g is almost a non-zero-divisor on M.*

Proof. Since $M' = (0 :_M g)$ is B-torsion if each element in M' is annihilated by some power of B. This means $M'_{\mathfrak{p}} = 0$ for all prime ideals $\mathfrak{p} \subset S$ and $B \not\subseteq \mathfrak{p}$. That is, g is a non-zero-divisor on the localization $M_{\mathfrak{p}}$. Therefore, we only need to show that g is not contained in any of the associated primes of M except possibly those which contain B. Let

$$\mathrm{Ass}_+(M) = \{\mathfrak{p} \in \mathrm{Ass}(M) : \mathfrak{p} \not\supseteq B\}.$$

When $n = 0$, we have $B = (0)$, $\mathrm{Ass}_+(M) = \emptyset$, and the assertion is trivial. Suppose $n > 0$, and $\mathrm{Ass}_+(M) = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_q\}$ where $\mathfrak{p}_i$ are G-graded ideals. Suppose we had an equality of R-modules:

$$S_{\mathbf{p}} = \max(R)S_{\mathbf{p}} \bigcup (\mathfrak{p}_1 \cap S_{\mathbf{p}}) \bigcup \cdots \bigcup (\mathfrak{p}_q \cap S_{\mathbf{p}})$$

Because R is an *um*-ring by theorem 6.4, $S_{\mathbf{p}}$ would have to be equal to one of the terms in the union. If $S_{\mathbf{p}} = \max(R)S_{\mathbf{p}}$, then by Nakayama's Lemma we must have

$S_{\mathbf{p}} = 0$. This is not the case: since $\mathbf{p}$ is in the Kaehler cone, we have by [13, Lemma 2.4] that

$$B \subseteq \sqrt{\langle S_{\mathbf{p}} \rangle}$$

which is nonzero since B is. Note that Lemma 2.4 of Maclagan-Smith's paper is valid here - the result is really about the combinatorics of monomials and has nothing to do with the ground ring R. If say $S_{\mathbf{p}} = \mathfrak{p}_1 \cap S_{\mathbf{p}}$, we would have $\langle S_{\mathbf{p}} \rangle \subseteq \mathfrak{p}_1$, and hence

$$B \subseteq \sqrt{\langle S_{\mathbf{p}} \rangle} \subseteq \sqrt{\mathfrak{p}_1} \subseteq \mathfrak{p}_1,$$

which contradicts the hypothesis on $\mathfrak{p}_1$. Any element g of

$$S_{\mathbf{p}} - \max(R) S_{\mathbf{p}} \bigcup (\mathfrak{p}_1 \cap S_{\mathbf{p}}) \bigcup \cdots \bigcup (\mathfrak{p}_q \cap S_{\mathbf{p}})$$

is almost a non-zero-divisor on M. □

Remark 6.6. Our main application of this section is to allow coarsening vector $\mathbf{v}$ as in Section 4, where $\mathbf{v}$ is not a positive coarsening vector. Thus some of the variables x_i might have $\mathbf{v} \cdot \deg(x_i) = 0$. Our method is to write $S = R[x_j]$ where $R = K[x_i]$ where x_i are the variables with $\mathbf{v} \cdot \deg(x_i) = 0$, and x_j are the variables with $\mathbf{v} \cdot \deg(x_j) > 0$, so that in effect the variables of $\mathbf{v}$-degree zero are treated as constants. The results of [19] refer to vanishing of local cohomology with respect to the ideal $\mathfrak{m}$ generated by all variables of S. If we allow some of the variables to have degree zero, then we must take $\mathfrak{m} = \langle x_j \rangle$, the ideal generated by the variables with strictly positive $\mathbf{v}$-degree. This is understood when applying the results of [19] to this case.

7. Examples

Example 7.1. Let $S = K[x_1, ..., x_n]$ be a polynomial algebra over a field with standard grading, $\deg(x_i) = 1$, $B = \langle x_1, ..., x_n \rangle$ and $\mathcal{C} = \{\mathbf{1}\}$. Let $m \geq 0$ be an integer and set

$$J = \{J_0 = m, J_1 = m+1, ...\}.$$

Then an easy calculation using Theorem 3.12 shows that $\mathrm{reg}_{B,\mathcal{C}}(J) = m + \mathbb{N}$. On the other hand let $\mathcal{D} = m + \mathbb{N}$, then Theorem 3.16 shows that $\mathrm{dreg}_{B,\mathcal{C}}(\mathcal{D}) = J$ for the set J above. These are exactly the regularity and degree regions that correspond to each other as in Theorem 1.1.

The next example is done in [19, Lemma 3.5]. We re-derive it using Theorem 3.12.

Example 7.2. Let $S = K[x_1, ..., x_n]$ be a polynomial algebra over a field with a weighted grading, $\deg(x_i) = a_i > 0$, $B = \langle x_1, ..., x_n \rangle$ and c is the least common multiple of the a_i. The Kaehler cone is $\mathcal{K} = \mathbb{N}c$. Take $\mathcal{C} = \mathcal{K}$. It is known that (see [13, Remark 3.3]) that $H^i_B(S) = 0$ unless $i = n$. Moreover, a computation based on [13, Proposition 3.2] shows that $H^n_B(S)_w = 0$ when $w > -\sum_{i=1}^n a_i$. Therefore

$$\begin{aligned}
\mathrm{reg}(S) &= \{u \in \mathbb{Z} : H^i_B(S)_w = 0,\ \forall i \geq 0,\ w \in u + \mathbb{N}c[1-i]\} \\
&= \{u \in \mathbb{Z} : H^n_B(S)_w = 0,\ \forall w \in u + \mathbb{N}c[1-n]\} \\
&= \{u \in \mathbb{Z} : H^n_B(S)_w = 0,\ \forall w \in u + (1-n)c + \mathbb{N}c\},
\end{aligned}$$

which shows that

$$\{u \in \mathbb{Z} : u \geq (n-1)c - \sum_{i=1}^{n} a_i + 1\} \subseteq \mathrm{reg}(S).$$

If M is a graded S-module with a minimal resolution of type J, then $\mathrm{reg}_{B,\mathcal{C}}(M)$ contains $\mathrm{reg}_{B,\mathcal{C}}(J)$ by Lemma 3.10 and this last one is seen by Theorem 3.12 to be the set of integers that simultaneously belong to all the sets

$$d_p + \mathrm{reg}(S) - pc + \mathbb{N}c, \quad \text{for all } d_p \in J_p.$$

Using the information on $\mathrm{reg}(S)$ above, we see that $\mathrm{reg}(M)$ contains the interval

$$\{m \in \mathbb{Z} : m \geq d_p + (n-p-1)c - \sum a_i + 1 \text{ for all } 0 \leq p \leq s,\ d_p \in J_p\}.$$

Example 7.3. Let $S = \mathbb{C}[x_{11}, \dots, x_{1m_1}; \dots ; x_{l1}, \dots x_{lm_l}]$, with $\deg(x_{ij}) = \mathbf{e}_i \in \mathbb{Z}^l = G$, where $\{\mathbf{e}_i\}$ is the standard basis of $\mathbb{R}^l$. This is the homogeneous coordinate ring of $\mathbf{P}^{m_1-1} \times \dots \times \mathbf{P}^{m_l-1}$. The ideal B is generated by all $x_{1,i_1} \dots x_{l,i_l}$. Set $\mathcal{C} = \{\mathbf{e}_i\}$. The theory developed in Section 5 applies, with $B_i = \langle x_{i1}, \dots, x_{im_i}\rangle$. For each $\emptyset \neq I \subseteq \{1, 2, \dots, l\}$, we define $B_I = \sum_{i \in I} B_i$. Define a vector grading for this family via $\mathbf{v}_I = \sum_{i \in I} \mathbf{e}_i$. We have

$$H^i_{B_I}(S)_{\mathbf{d}} = 0, \text{ for all } \mathbf{d} \cdot \mathbf{v}_I \geq 1 - i, \text{ for each } I.$$

This implies

$$\mathrm{reg}_{B_*,\mathcal{C},\mathbf{v}_*}(S) = \mathbb{N}^l.$$

Let M be a $\mathbb{Z}^l$-graded S-module with minimal resolution

$$0 \longrightarrow \bigoplus_{\mathbf{d}_s \in J_s} S(-\mathbf{d}_s) \longrightarrow \cdots \longrightarrow \bigoplus_{\mathbf{d}_0 \in J_0} S(-\mathbf{d}_0) \longrightarrow M \longrightarrow 0$$

If $\mathbf{p} \in \mathrm{reg}_{B_*,\mathcal{C},\mathbf{v}_*}(M)$ then the degrees of the syzygies satisfy the bounds $\mathbf{d}_i \cdot \mathbf{v}_I \leq \mathbf{p} \cdot \mathbf{v}_I + i$ for all $\emptyset \neq I = \{1, 2, \dots, l\}$. (compare [12, Section 4]). This can be seen as an application of Proposition 5.7. First note that in the Definition 5.5 we have $m(1, i) = 1$ for all i. Therefore if $\mathbf{p} \in \mathrm{reg}_{B_*,\mathcal{C},\mathbf{v}_*}(M)$ the condition

$$H^i_{B_I}(M)_{\mathbf{d}} = 0 \text{ for all } \deg_{\mathbf{v}_I}(\mathbf{d}) \geq \deg_{\mathbf{v}_I}(\mathbf{m}) + (1 - i)$$

shows that $b_I := \mathbf{v}_I \cdot \mathbf{p} \geq \text{reg-num}_{\mathbf{v}_\mathbf{I}}(M)$ since $\text{reg-num}_{\mathbf{v}_\mathbf{I}}(M)$ is the least integer which gives the above vanishing statement for all $i \geq 0$. Since $c_{\mathbf{v}_I} = 1$, $\deg_{\mathbf{v}_I}(\mathbf{p}) = \mathbf{v}_I \cdot \mathbf{p}$ satisfies the hypotheses of Proposition 5.7. From that proposition and Corollary 4.6 we conclude that the ith syzygies are in $K_i(\mathbf{v}_*, b_*)$.

Note that $s_{\mathbf{v}_I} = 1$, so that this region is just $\mathbf{d}_i \cdot \mathbf{v}_I \leq \mathbf{p} \cdot \mathbf{v}_I + i$.

Example 7.4. Let $S = \mathbb{C}[x_1, x_2, x_3, x_4]$ be the coordinate ring of the Hirzebruch surface $\mathbb{F}_t$ with $t \geq 0$, $G = \mathbb{Z}^2$, $\deg(x_1) = (1, 0)$, $\deg(x_2) = (-t, 1)$, $\deg(x_3) = (1, 0)$, and $\deg(x_4) = (0, 1)$. $B = (x_1x_2, x_1x_4, x_2x_3, x_3x_4)$. Let $\mathcal{C} = \{(1, 0), (0, 1)\}$. Then ([13, Example 4.3])

$$\mathrm{reg}(S) = \begin{cases} \mathbb{N}^2 & \text{for } t = 0, 1; \\ ((t-1, 0) + \mathbb{N}^2) \cup ((0, 1) + \mathbb{N}^2) & \text{for } t \geq 2. \end{cases}$$

Natural choices of vector coarsenings are $\mathbf{v} = (0, 1)$ and $\mathbf{w} = (1, t)$. The $\mathbf{v}$-gradings behave just like Example 7.1 but with the variables x_1 and x_3 assigned degree 0. The $\mathbf{w}$-gradings behave just like Example 7.2 the variable x_2 assigned degree 0, the variables x_1 and x_3 given degree 1 and the variable x_4 given degree t. By imposing

regularity conditions with respect to these vector gradings we can describe syzygy regions by inequalities for some $a,\ b,\ c,\ d$:

$$\begin{aligned}\mathbf{d}_i\cdot\mathbf{v} &\leq \mathbf{p}\cdot\mathbf{v}+ai+b,\\ \mathbf{d}_i\cdot\mathbf{w} &\leq \mathbf{p}\cdot\mathbf{w}+ci+d.\end{aligned}$$

Let $Z=\mathbb{V}(B)=Z_1\cup Z_2=\mathbb{V}(B_1)\cup\mathbb{V}(B_2)$, where $B_1=(x_1,x_3)$, $B_2=(x_2,x_4)$. This is the Batryev decomposition in this case. However, there is not an orthogonal choice of vector gradings for the ideals B_1, B_2 in the sense of Section 5 so it does not seem that we can apply the results of that section here.

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